

**Problem Set IV: Due Thursday, November 19, 2015**

- 1.) Use a reduced form of a Variational Principle discussed in class to show that Action is an adiabatic invariant for an oscillator with slowly varying parameter. Discuss the connection of this exercise to the WKB calculation discussed in class.
- 2.) What is the relation between the volume and pressure of a gas of point particles moving within a cubic box of side  $L$ ? Assume the walls are hard, so particles reflect elastically, and that two opposite sides move together or apart *slowly*. Comment on the relation between your result and those of thermodynamics.
- 3.) In the course so far, we have discussed *three* types of multiple time scale approximation methods in mechanics. Give a concise summary of these. Include a table. Your summary should include:
  - a listing of the disparate time scales
  - the approximation made
  - why it works – what is the leverage
  - any key features – i.e. resonances, etc.
  - the ‘canonical example’ for each
  - a one line summary of the bottom line for the canonical example

N.B.: For those who are puzzled at this sort of assignment, see: “The Innovators”, by Walter Isaacson; location 1626 re: Grace Hopper.

- 4a.) Determine the stable equilibrium positions for a simple pendulum which oscillates:
  - a.) horizontally, with  $x = x_0 \cos \omega t$
  - b.) in a circle, with  $x = r_0 \cos \omega t$ ,  $y = r_0 \sin \omega t$ .

Take  $\omega \gg \sqrt{g/\ell}$  and consider the full range of parameters.

- 5.) Now again consider a simple pendulum with support oscillating at  $y = y_0 \cos \omega t$ . If the pendulum has length  $\ell$  (so  $\omega_0 = \sqrt{g/\ell}$ ) and  $\omega = 2\omega_0 + \epsilon$ , determine the conditions for, and growth rate of, parametric instability.
- 6a.) Compute the threshold for parametric instability in the presence of linear frictional damping, as well as mismatch. For what range of mismatch  $\epsilon$  will instability occur?

7.) Let  $H(q, p, t) = H_0(q, p) + V(q)d^2A/dt^2$  where  $A(t)$  is periodic, with period  $\tau \ll T$ . Here  $T$  is the period of the motion governed by  $H_0$ .

- a.) Derive the mean field (i.e. short time averaged) equations for this system.
- b.) Show that these mean field equations may be obtained from the effective Hamiltonian

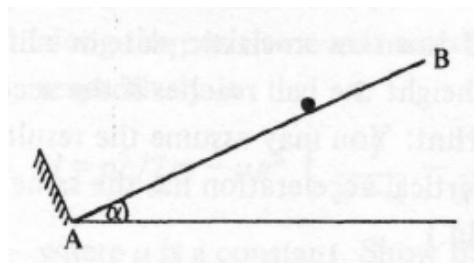
$$K(p, q) = H_0(p, q) + \frac{1}{4m} \left\langle \left( \frac{dA}{dt} \right)^2 \right\rangle \left( \frac{\partial V(q)}{\partial q} \right)^2.$$

Here  $\langle \rangle$  means a short time average. You may assume  $H_0 = p^2/2m + V_0(q)$ .

8.) An ice skater S is freely swung around a fixed point, 0, on horizontal smooth ice. Write down the Hamiltonian if the distance  $R(t)$  from 0 to S is a given function of  $t$ .

- a.) Use the principle of adiabatic invariance to show that if  $R(t)$  changes slowly from  $R_0$  to  $\alpha R_0$ , then the skater's kinetic energy changes from  $E_0$  to approximately  $E_0/\alpha^2$ .
- b.) Show that in this example the action is an exact invariant. In the particular case  $R(t) = (1 + \epsilon t) R_0$ , where  $\epsilon$  is a small positive constant, find the motion explicitly and verify that the above approximation to the change in the kinetic energy is accurate to order  $\epsilon$  for all time.

9.) A particle moves smoothly up and down an inclined plane AB, as shown in the diagram, and is elastically reflected by a wall at A. How does the maximum height the particle reaches change when the angle  $\alpha$  changes slowly?



10.) Sometimes the action variable is *defined* to be equal to the area under the phase curve. Show that, with this definition, the conjugate variable increases by unity during each period.

- 11.) Find the angle-action variables for a particle of mass  $m$  rotating in the periodic potential

$$V(\psi) = A\psi \quad (0 \leq \psi \leq \alpha < \pi)$$

$$= A\alpha \quad (\alpha \leq \psi \leq \pi)$$

$$V(-\psi) = V(+\psi)$$

where  $A$  and  $\alpha$  are positive constants. Determine the frequency of the motion of energy  $E$ .